Lecture Notes, Wednesday, January 18, 2012

7. Functions

We describe a function f(•) as follows:
For each $x \in A$ there is $y \in B$ so that $y = f(x)$.
$f: A \to B$.
A = domain of f
B = range of f
graph of f = S ⊂ $A \times B$, S = {(x, y) | y = f(x)}
Let T⊂A. f(T) ≡ {y | y = f(x), x ∈T} is the image of T under f.
$f^{-1}: B \to A$, f^{-1} is known as "f inverse"
$f^{-1}(y) = \{x \mid x \in A,\ y = f(x)\}$

7.1 Continuous Functions

Let $f : A \to B,\ A \subset R^m,\ B \subset R^p$.

The notion of continuity of a function is that there are no jumps in the function values. Small changes in the argument of the function (x) result in small changes in the value of the function (y=f(x)).

Let ε, δ(ε), be small positive real numbers; we use the functional notation δ(ε) to emphasize that the choice of δ depends on the value of ε. f is said to be **continuous** at a ∈ A if

(i) for every ε > 0 there is δ(ε) > 0 such that
$|x - a| < \delta(\varepsilon) \Rightarrow |f(x) - f(a)| < \varepsilon$, or equivalently,

(ii) $x^\nu \in A,\ \nu = 1, 2, \ldots,$ and $x^\nu \to a,$ implies $f(x^\nu) \to f(a)$.

Theorem 7.5: Let $f : A \to B$, f continuous. Let $S \subset B$, S closed. Then $f^{-1}(S)$ is closed.
Proof: Let $x^\nu \in f^{-1}(S)$, $x^\nu \to x^o$. We must show that $x^o \in f^{-1}(S)$. Continuity of f implies that
$f(x^\nu) \to f(x^o)$. $f(x^\nu) \in S$, S closed, implies $f(x^o) \in S$. Thus $x^o \in f^{-1}(S)$. QED

Note that as a consequence of Thm 7.5, the inverse image under a continuous function of an open subset of the range is open (since the complement of a closed set is open).

Theorem 7.6: Let $f : A \to B$, f continuous. Let $S \subset A$, S compact. Then f(S) is compact.

Proof: We must show that f(S) is closed and bounded.
Closed: Let $y^\nu \in f(S)$, $\nu=1,2,...$, $y^\nu \to y^o$. Show that $y^o \in f(S)$. There is $x^\nu \in S$, $x^\nu = f^{-1}(y^\nu)$. Take a convergent subsequence, relabel, and $x^\nu \to x^o \in S$ by closedness of S. But continuity of f implies that $f(x^\nu) \to f(x^o) = y^o \in f(S)$.

Bounded: For each $y \in f(S)$, let $C(y)=\{z| z\in B, |y-z|<\varepsilon\}$, an ε-ball about y. The family of sets $\{f^{-1}(C(y))| y\in f(S)\}$ is an open cover of S (the inverse image of an open set under f is open since the inverse image of its complement --- a closed set --- is closed, Thm 2.6). There is a finite subcover. Hence f(S) is covered by a finite family of ε balls. f(S) is bounded. QED

Corollary 7.2: Let $f : A \to R$, f continuous, $S \subset A$, S compact, then there are $\bar{x}, \underline{x} \in S$ such that $f(\bar{x}) = \sup\{f(x)| x \in S\}$ and $f(\underline{x}) = \inf\{f(x)| x \in S\}$, where inf indicates greatest lower bound and sup indicates least upper bound.

Corollary 7.2 is very important for economic analysis. It provides sufficient conditions so that maximizing behavior takes on well defined values.

Homogeneous Functions

$f: R^p \to R^q$.
f is homogeneous of degree 0 if for every scalar (real number) $\lambda > 0$, we have $f(\lambda x) = f(x)$.
f is homogeneous of degree 1 if for every scalar $\lambda > 0$, we have $f(\lambda x) = \lambda f(x)$.